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Controllability of cross-flow heat exchangers

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Abstract

The operation of heat exchangers and other thermal equipment in the face of variable loads is usually controlled by manipulating inlet fluid temperatures or mass flow rates. This is a fundamental study of the basic issues regarding state and output controllability in such systems. A numerical method based on finite-differences is developed to approximate infinite-dimensional equations by finite-dimensional ones for the study of a conduction–convection system. The dynamics of a single-pass cross-flow heat exchanger with simultaneous advection, convection and conduction, in which water and air are the in- and over-tube fluids, respectively, is represented by a coupled set of partial differential equations. The numerical method is used to analyze the behavior of the heat exchanger equations. Using the water or air inlet temperature as the manipulated variable leads to a linear problem, and for the water flow rate it is non-linear. Controllability results for different choices of the manipulated variable are presented.

1. Introduction

Before attempting to design a control strategy for a system to achieve a desired objective, it is clearly desirable to determine whether any control is possible for the system. This can be done by investigating its *state controllability*; this is the ability of the complete system to be taken from any given state to any other within a prescribed time interval. *Output controllability* is a similar concept applied only to the output of the system. In thermal systems controllability usually means that a system with an initial temperature distribution is able to move to any other in finite time by means of a suitable control input which could be in the form of a flow rate, an applied heat flux or an externally applied temperature.

Controllability is easily tested for systems governed by a system of linear, finite-dimensional ordinary differential equations [1]. The situation is more complicated for linear infinite-dimensional systems such as those governed by partial differential equations (PDEs) [2]. Controllability is exact if the function representing the state can be taken from an initial to a final target state, and is approximate if it can be taken to a neighborhood of the target [3]. Determination of approximate controllability is usually sufficient, and is the goal here since it makes sense in most engineering problems. Although an uncontrollable thermal system cannot in general be taken from any state to any other, it is for many applications not necessary since state controllability may be less important than output controllability. For example, Rosenbrock [4] notes that most industrial plants are controlled quite satisfactorily though they are not state controllable. Constrained controllability where the manipulated inputs such as flow rates and temperatures have finite bounds is also very relevant to thermal engineering.

Controllability for different kinds of dynamical systems governed by PDEs has been considered in many publications (see [5] for an extensive list). Applications to thermal problems, however, are very limited, though there has been some work in the areas of industrial and chemical plants and thermal networks [6]. The controllability of multi-stream heat exchangers, when some

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Nomenclature

A	matrix operator for system
A_c	cross-sectional area [m^2]
$\mathscr{A}$	operator for system
$\mathscr{B}$	operator for manipulated variable
B	matrix operator for manipulated variable
C	matrix operator for output variable
C_M	condition number of controllability matrix
c	specific heat [J/kg K]
h	heat transfer coefficient [W/m^2 K]
k	thermal conductivity [W/m K]
L	length [m]
$\dot{m}$	mass flow rate [kg/s]
M	system controllability matrix
n	number of finite-difference divisions
N	output controllability matrix
P	perimeter [m]
r	radius [m]
t	time [s]
T	temperature [°C]
T_L	boundary temperature [°C]
u	manipulated variable
W^{-1}	reachability grammian
x	spatial coordinate [m]
Δx	grid spacing
y	output variable

Greek symbols

α	thermal diffusivity [m^2/s]
β	eigenvalue
ζ	parameter representing convection [s^{-1}]
θ	temperature variable [°C]
ρ	density [kg/m^3]
σ	$= \alpha/\Delta x^2$ [s^{-1}]
ϕ	eigenfunction

Subscripts and superscripts

a	air
f	final
i	inner
in	inlet
o	outer
out	outlet
t	tube
T	transpose
w	water
∞	ambient

operating parameters deviate from their design value, has also been studied recently [7].

In this paper the controllability of a single-pass, single-circuit, cross-flow heat exchanger that has been previously studied both experimentally [8] and numerically [9] will be investigated. The mathematical model is a pair of coupled partial differential equations representing conduction in the tube wall, advection by the in-tube fluid, and lateral convection to the over-tube fluid. An approach based on finite-differences will be developed and tested on a simpler problem with only conduction and convection. This approach is convenient for several reasons: it reduces the problem to one of finite dimensions for which application of existing theory is straightforward, it can be easily applied to operators that are not self-adjoint, and it can also be conveniently used in other thermal and fluid mechanical problems in which large-scale numerical methods are currently used for flow computations. Control by manipulating inlet temperatures will be analyzed in this way.

The problem of flow rate manipulation, which is also very common in thermal systems, is special; it is nonlinear for which general controllability theorems are not available. However, it turns out that due to the physics of the phenomena that take place, controllability within definite bounds can also be demonstrated.

2. Diffusive-convective system

In this section the fin equation, which is a one-dimensional conduction–convection system that gives a single second-order PDE [10], will be analyzed. This has many of the aspects to be considered later in the heat exchanger model. Though the controllability of this system has been analyzed previously [5], it will be shown that it can be studied using either infinite-or finite-dimensional approaches.

A conductive bar of length L that is being cooled or heated from the side as schematically shown in Fig. 1 is considered. There is conduction along the bar as well as convection to the surroundings from the side. The temperature distribution is governed by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - \zeta (T - T_\infty), \tag{1}$$

where $T(x,t)$ is the temperature distribution along the bar representing the state of the system, T_∞ is the temperature of the surroundings, t is time, and x is the longitudinal coordinate measured from one end. The thermal diffusivity is α, and $\zeta = hP/\rho c A_c$ where h is the convective heat transfer coefficient, A_c is the constant cross-sectional area of the bar, P is the perimeter of the cross-section, ρ is the density, and c is the specific heat.

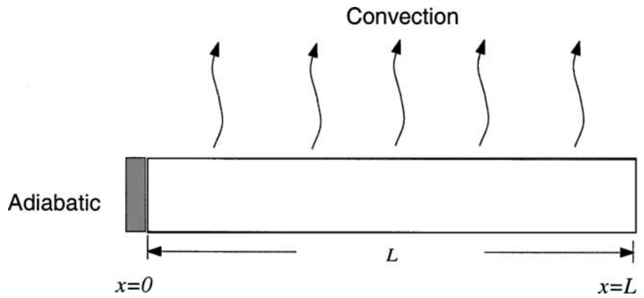

Fig. 1. One-dimensional convection-conduction heat transfer problem.

For simplicity it will be assumed that ζ is independent of x.

The system is assumed to be initially at a uniform temperature. This does not imply any loss of generality since if a linear system is indeed controllable it can be taken from any state to any other. An adiabatic condition at the end $x = 0$ will be assumed so that $(\partial T/\partial x)(0,t) = 0$. Either the surrounding temperature T_∞ or the temperature of the other end $T(L,t)$ can be used as a manipulation variable for control purposes. In this context these two single-input methods are known as distributed and boundary control since the manipulated variable enters through the equation and the boundary condition, respectively. They will be analyzed separately.

2.1. Distributed control

In this section the controllability of the system will be analyzed in two different ways: as a continuous system and using a finite-dimensional approximation. The manipulated variable is the ambient temperature $T_\infty(t)$ with a fixed boundary condition $T(L,t) = T_L$. Using T_L as a reference temperature and defining $\theta = T - T_L$, Eq. (1) becomes,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} - \zeta\theta + \zeta\theta_\infty(t) \tag{2}$$

with the homogeneous boundary and initial conditions $(\partial\theta/\partial x)(0,t) = 0$, $\theta(L,t) = 0$, and $\theta(x,0) = 0$. For convenience, the initial temperature distribution has been taken to be zero.

2.1.1. Continuous system

Consider a system governed by

$$\frac{\partial \theta}{\partial t} = \mathscr{A}\theta + \mathscr{B}u \tag{3}$$

with suitable boundary and initial conditions, where $\mathscr{A}$ is a bounded semi-group operator, $\mathscr{B}$ is a linear operator, and $u(t)$ is the manipulated variable. $\mathscr{A}$ operates on elements of a vector space of functions that satisfy the homogeneous spatial boundary conditions. If $\mathscr{A}$ is self-adjoint, then it has real eigenvalues β_m, with $m = 0, 1, 2, \ldots$, and a complete orthonormal set of eigenfunctions $\phi_m(x)$ which forms a spatial basis for θ. It is known [5] that the system is state controllable if and only if all the inner products

$$\langle \mathscr{B}, \phi_m \rangle = \int_0^L \mathscr{B}\phi_m \,\mathrm{d}x \neq 0. \tag{4}$$

In the present case

$$\mathscr{A} = \alpha \frac{\partial^2}{\partial x^2} - \zeta,$$

$$\mathscr{B} = \zeta,$$

$$u = \theta_\infty.$$

The eigenvalues and eigenfunctions are

$$\beta_m = -\frac{(2m+1)^2\pi^2}{4L^2} - \zeta,$$

$$\phi_m = \sqrt{\frac{2}{L}} \cos\frac{(2m+1)\pi x}{2L}.$$

Inequality (4) is satisfied for all m, so the system is indeed state controllable.

2.1.2. Finite-dimensional approximation

Though the continuous-systems approach worked for this simple problem, it is desirable to develop a numerical approximation for the controllability test which can also be used for more complicated problems. By dividing the domain $[0,L]$ into n equal parts of size Δx, a finite-difference spatial discretization of Eq. (2) gives

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = -(2\sigma + \zeta)\theta_i + \sigma(\theta_{i-1} + \theta_{i+1}) + \zeta\theta_\infty,$$

where $\sigma = \alpha/\Delta x^2$. The nodes are $i = 1, 2, \ldots, n+1$, where $i = 1$ is at the left and $i = n+1$ at the right end. The boundary conditions used at the two ends are $\theta_0 = \theta_1$ and $\theta_{n+1} = 0$, respectively. Collecting the equations for all the nodes

$$\frac{d\theta}{dt} = A\theta + Bu, \tag{5}$$

where

$$\theta(t) = [\theta_1, \theta_2, \ldots, \theta_n]^{\mathrm{T}} \in \mathbb{R}^n \tag{6}$$

and $u(t) = \theta_\infty \in \mathbb{R}$. Also

$$A = \begin{bmatrix} -(2\sigma+\zeta) & 2\sigma & 0 & \cdots & 0 \\ \sigma & -(2\sigma+\zeta) & \sigma & & \vdots \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & & & \sigma \\ 0 & \cdots & 0 & \sigma & -(2\sigma+\zeta) \end{bmatrix} \in \mathbb{R}^{n\times n},$$
$$B = \zeta[1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n, \tag{7}$$

where the boundary conditions have been applied to make A non-singular.

It is known [1] that the state of a system of the form of Eq. (5) is completely controllable if and only if the matrix

$$M = [B, AB, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n} \tag{8}$$

is of full rank. In the present case it can be shown that

$$\det\ M = (-1)^{\lfloor n/2 \rfloor}\sigma^{n(n-1)/2}\zeta^n,$$
$$\operatorname{rank}\ M = n,$$

where $\lfloor\cdot\rfloor$ is the floor function. The matrix M is of full rank, indicating that the state of the system is controllable, a conclusion that is also obtained from the eigenfunction expansion. This lends support to the use from now on of the finite-difference approximation to analyze controllability.

2.2. Boundary control

Here the boundary condition $T(L,t) = T_L(t)$ will be the manipulated variable through which control is exercised. Using the constant outside temperature T_∞ as reference and defining $\theta = T - T_\infty$, Eq. (1) becomes,

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2} - \zeta\theta \tag{9}$$

with the initial and boundary conditions $(\partial\theta/\partial x)(0,t) = 0$, $\theta(L,t) = T_L(t) - T_\infty$, and $\theta(x,0) = 0$.

2.2.1. State controllability

Eq. (9) can be discretized to take the form of Eq. (5), where $\theta(t)$ is given by Eq. (6), A is given by Eq. (7), $u(t) = \theta(L,t) \in \mathbb{R}$ and

$$B = [0, \ldots, \sigma]^{\mathrm{T}} \in \mathbb{R}^n. \tag{10}$$

At the left end the adiabatic condition is the same as before. However, the temperature at right end θ_{n+1} is not known but is the manipulated variable u.

The controllability matrix M is

$$M = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \sigma^n \\ 0 & \cdots & 0 & \sigma^{n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \sigma^3 & \cdots & \cdots \\ 0 & \sigma^2 & -2\sigma^2(2\sigma+\zeta) & \cdots & \cdots \\ \sigma & -\sigma(2\sigma+\zeta) & \sigma^3+\sigma(2\sigma+\zeta)^2 & \cdots & \cdots \end{bmatrix}$$

so that now

$$\det\ M = (-2)^{\lfloor n/2 \rfloor}\sigma^{(n^2+n)/2},$$
$$\operatorname{rank}\ M = n.$$

Thus M is again of full rank, indicating that the state of the system is boundary controllable.

2.2.2. Output controllability

Up to now control of the complete state of the system has been considered. In thermal systems, however, it is unusual to be able to observe the complete temperature distribution. Most of the times users are interested in or able to work with only a vector $y \in \mathbb{R}^p$, called the output, where

$$y(t) = C\theta(t) \tag{11}$$

with $C \in \mathbb{R}^{p\times n}$. Output controllability refers to the ability of a suitable control input $u(t)$ to be able to take the output $y(t)$ from any point to any another. The system represented by Eqs. (5) and (11) is output controllable [11] if and only if the rank of the matrix

$$N = [CB, CAB, \ldots, CA^{n-1}B] \in \mathbb{R}^{p\times n}$$

is p. If, for example, it is desired to control the temperature at $x = 0$ which is θ_1, then

$$C = [1, 0, \ldots, 0] \in \mathbb{R}^{1\times n}.$$

Thus

$$N = [0, \ldots, 0, 2\sigma^n]$$

which has a rank equal to $p = 1$ indicating that this output is controllable.

2.2.3. Optimal control

Since the system is boundary controllable, there exists a control function $T_L(t)$ which transfers the system from the initial state $\theta_0 = \theta(x,0)$ to the target state $\theta_{\mathrm{f}} = \theta(x,t_{\mathrm{f}})$ within a finite time t_{f}. The solution of Eq. (9) is

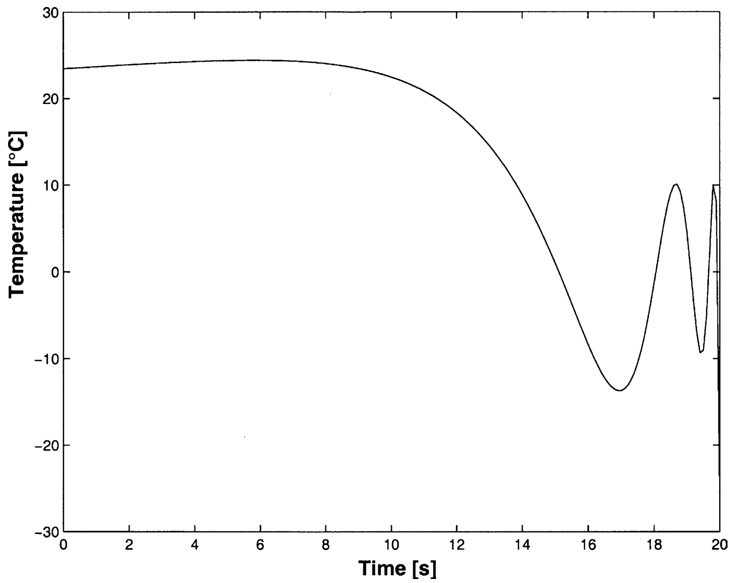

Fig. 2. Variation of T_L with time for optimal boundary control.

$$\theta(x,t) = \mathrm{e}^{-\zeta t}\left[\frac{2\alpha}{L}\sum_{m=0}^{\infty}\gamma_m \mathrm{e}^{-\alpha\gamma_m^2 t}(-1)^m \cos\gamma_m \times \int_{t'=0}^{t} \mathrm{e}^{\alpha\gamma_m^2 t'}[T_L(t') - T_\infty]\,\mathrm{d}t'\right],$$

where $\gamma_m = (2m+1)\pi/2L$. Finding $T_L(t)$ requires the solution of this Fredholm integral equation of the first kind, and there are ways to solve it numerically [12]. However, it is obvious that the control input, i.e. the temperature at the boundary, is not unique.

With a finite-dimensional approximation, however, optimal control theory [1] can be used to get

$$u(t) = \theta(L,t) = B^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(t-t_{\mathrm{f}})}W^{-1}(0,t)[\theta_0 - \mathrm{e}^{At}\theta_{\mathrm{f}}],$$

where $\mathrm{e}^{At} = \sum_{k=0}^{\infty}(t^k/k!)A^k$. θ_0 and θ_{f} are n-dimensional vectors representing the initial and target temperatures, respectively. $W^{-1}(0,t)$ is a $n \times n$ matrix called the reachability gramian of the system defined by

$$W(0,t) = \int_0^t \mathrm{e}^{(s-t_{\mathrm{f}})A}BB^{\mathrm{T}}\mathrm{e}^{(s-t_{\mathrm{f}})A^{\mathrm{T}}}\,\mathrm{d}s.$$

Thus, starting from a zero temperature distribution any given temperature distribution in the finite-dimensional approximation can be reached at a given time by varying only the temperature at one end of the bar.

As a special case, if it is desired that the output of the system be the temperature distribution along the bar, then C in Eq. (11) is the identity matrix. This, of course, is state controllability which has already been confirmed. For example, the system can be discretized in a small number of divisions, say $n = 6$, and it may be required that the temperatures at these locations be changed from the initial values of (say) $\theta = 0$ °C to the final temperature of $\theta_{\mathrm{f}} = 25$ °C in a time interval $t_{\mathrm{f}} = 20$ s. For illustration, numerical values of the parameters are set at $\zeta = 0.0118$ s^{-1} and $\sigma = 1$ s^{-1}. Matrices M and N are equal and are found to have a rank equal to 6, indicating that the system is controllable. The results of optimal control are shown in Figs. 2 and 3. Fig. 2 shows the variation of the temperature at the end of the bar that will take the temperature distribution to the target. In Fig. 3 the variation of the temperature with time at each of the six nodes is plotted.

3. Cross-flow heat exchanger model

Among the many kinds of water-to-air heat exchangers, the cross-flow geometry is very common. Though they sometimes have multiple rows and circuits, the simplest geometry that can be easily computed, i.e. a single-tube with water flow inside and cross-flow of air outside, will be considered here. A schematic of this arrangement is shown in Fig. 4. Although a straight tube is shown, it may zig-zag over the face of the heat exchanger so as to make it more compact. Controllability analysis based on a finite-difference approximation of the governing equations will be carried out.

To enable a one-dimensional analysis, the simplifying assumptions that the flow is hydrodynamically and thermally fully developed, and that the velocity and temperature are uniform over the cross-section of the pipe will be used. The physical properties of the fluid are

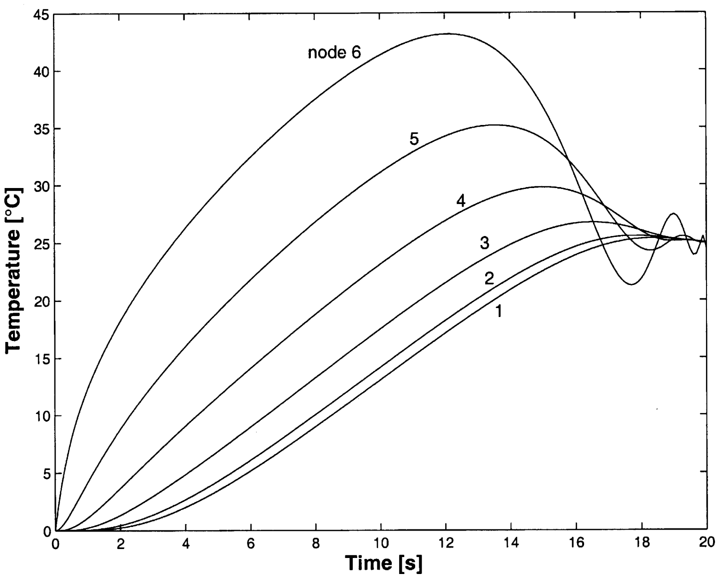

Fig. 3. Variation of the temperatures at the six nodes with time as a result of optimal boundary control. The nodes are numbered sequentially from the $x = 0$ end.

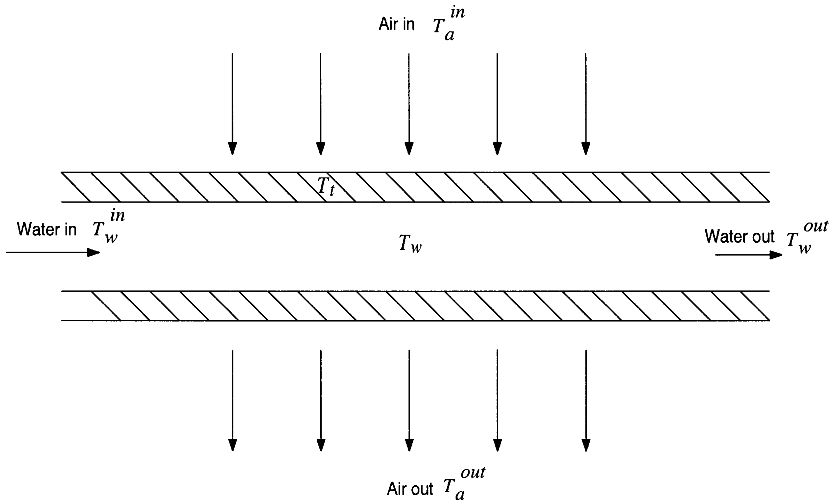

Fig. 4. Schematic of single-tube cross-flow heat exchanger with in-tube water, tube wall and over-tube air.

also constant. There is convective heat transfer between the water and the tube wall, conduction along the tube wall, and convection between the tube wall and the surrounding air. The heat transfer coefficients are the same as in [9]. The following are the governing equations for this problem with appropriate boundary conditions.

On the outside of the tube

$$\frac{\dot{m}_a}{L} c_a (T_a^{in} - T_a^{out}) = h_o 2\pi r_o (T_a - T_t), \tag{12}$$

where L is the length of the tube, $\dot{m}_a$ is the mass flow rate of air, c_a is its specific heat, T_a^{in} and T_a^{out} are the incoming and outgoing air temperatures, h_o is the heat transfer coefficient in the outer surface of the tube, r_o is the outer radius of the tube, T_a is the air temperature surrounding the tube, and T_t is the tube wall temperature. For convenience, the air temperature can be assumed to be approximately

$$T_a = \frac{T_a^{in} + T_a^{out}}{2}. \tag{13}$$

This can be substituted in Eq. (12).

In the water

$$\rho_w c_w \pi r_i^2 \frac{\partial T_w}{\partial t} + \dot{m}_w c_w \frac{\partial T_w}{\partial x} = h_i 2\pi r_i (T_t - T_w), \tag{14}$$

where ρ_w is the water density, c_w is its specific heat, and $\dot{m}_w$ is the water mass flow rate. In these equations, in general $T_t = T_t(x,t)$, $T_w = T_w(x,t)$, $T_a^{out} = T_a^{out}(t)$, and $T_a = T_a(t)$. The boundary and initial conditions are $T_t(0,t) = T_w(0,t) = T_w^{in}$, $T_t(L,t) = T_w(L \cdot t)$, and $T_t(x,0) = T_w(x,0) = 25$ °C (arbitrarily).

Finally, in the wall of the tube

$$\rho_t c_t \pi (r_o^2 - r_i^2) \frac{\partial T_t}{\partial t} = k_t \pi (r_o^2 - r_i^2) \frac{\partial^2 T_t}{\partial x^2} + 2\pi r_o h_o (T_a - T_t) - 2\pi r_i h_i (T_t - T_w), \tag{15}$$

where ρ_t is the density of the tube material, c_t is its specific heat, k_t is its thermal conductivity, r_i is the inner radius of the tube, h_i is the heat transfer coefficient in the inner surface of the tube, and T_w is the water temperature.

Apart from geometry and material properties, there are four parameters in Eqs. (12)–(15): two mass flow rates and two inlet temperatures for water and air, respectively, that can be used for control purposes as the manipulated variable. Two single-input cases will be analyzed for the purpose of controlling the fluid outlet temperatures. First, when the mass flow rates $\dot{m}_a$ and $\dot{m}_w$ are constant and control of the heat exchanger outlet temperatures T_a^{out} and T_w^{out} is accomplished by manipulating either T_a^{in} or T_w^{in}. Second, when one of the flow rates ($\dot{m}_w$ will be used as an example) is used as a manipulated variable; the problem is then non-linear.

4. Manipulated variable: water inlet temperature

In this section the water inlet temperature T_w^{in} will be used as a manipulated variable while all other parameters like T_a^{in}, $\dot{m}_w$, and $\dot{m}_a$ are constant.

4.1. Finite-dimensional approximation

Dividing the computational domain spatially into n parts and using finite-differences, Eqs. (12)–(15) can be put in the form of Eq. (5). This is done by approximating first- and second-order derivatives by upwind and central differences, respectively, for Eqs. (15) and (14), and eliminating the algebraic Eq. (12) to give

$$\frac{dT_t}{dt} = a_1 T_{t,i} + c_t (T_{t,i+1} + T_{t,i-1}) + a_2 T_{w,i} + a_3 T_a^{in}, \tag{16}$$

$$\frac{dT_w}{dt} = -b_1 T_{w,i} + b_2 T_{w,i-1} + a_4 T_{t,i}, \tag{17}$$

where

$$a_1 = \frac{2\pi r_o h_o}{\rho_t c_t \pi (r_o^2 - r_i^2)} \left(\frac{a_5}{2 + a_5} - 1 \right) - \frac{2\alpha_t}{\Delta x^2} - a_2,$$

$$a_2 = \frac{2\pi r_i h_i}{\rho_t c_t \pi (r_o^2 - r_i^2)},$$

$$a_3 = \frac{2\pi r_o h_o}{\rho_t c_t \pi (r_o^2 - r_i^2)} \left(\frac{1}{2} + \frac{1 - a_5/2}{2 + a_5} \right),$$

$$a_4 = \frac{4h_i}{\rho_w c_w D_i},$$

$$a_5 = \frac{2\pi r_o h_o L}{\dot{m}_a c_a},$$

$$b_1 = \frac{\dot{m}_w}{\rho_w \pi r_i^2 \Delta x} + a_4,$$

$$b_2 = \frac{\dot{m}_w}{\rho_w \pi r_i^2 \Delta x},$$

$$c_t = \frac{\alpha_t}{\Delta x^2}.$$

The boundary conditions are

$$T_{t,0} = T_{w,0} = T_w^{in}, \tag{18}$$

$$T_{t,n} = T_{w,n}. \tag{19}$$

In the following the variables

$$T(t) = [T_{t,1}(t), T_{w,1}(t), T_{t,2}(t), T_{w,2}(t), \ldots, T_{w,n}(t)]^T \in \mathbb{R}^{(2n-1)\times 1}, \tag{20}$$

$$A = \begin{bmatrix} a_1 & a_2 & c_t & 0 & \cdots & & & & & & 0 \\ a_4 & -b_1 & 0 & \cdots & & & & & & & 0 \\ c_t & 0 & a_1 & a_2 & c_t & 0 & \cdots & & & & 0 \\ 0 & b_2 & a_4 & -b_1 & 0 & \cdots & & & & & 0 \\ 0 & 0 & c_t & 0 & a_1 & a_2 & c_t & 0 & \cdots & & 0 \\ 0 & 0 & 0 & b_2 & a_4 & -b_1 & 0 & \cdots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & & & & & & \\ 0 & \cdots & & & & & & 0 & b_2 & a_4 & -b_1 \end{bmatrix} \in \mathbb{R}^{(2n-1)\times(2n-1)} \tag{21}$$

will be used. B will depend on the choice of the manipulated variable.

4.2. State controllability

The governing equations can be written as

$$\frac{dT}{dt} = AT + Bu + F, \tag{22}$$

where $T(t)$ and A are given by Eqs. (20) and (21), and

$$B = \left[\frac{\alpha_t}{\Delta x^2}, \frac{\dot{m}_w}{\rho_w \pi r_i^2 \Delta x}, 0, \ldots, 0\right]^T \in \mathbb{R}^{2n-1\times 1},$$

$$F = [a_3, 0, a_3, 0, \ldots, a_3, 0]^T T_a^{in} \in \mathbb{R}^{2n-1\times 1},$$

$$u = T_w^{in}.$$

Since A is non-singular the transformation

$$\theta = A^{-1}F + T \tag{23}$$

can be introduced to write Eq. (22) in the form of Eq. (5). For any n, it can be shown by computation that the controllability matrix M defined in Eq. (8) has a full rank, indicating that θ is controllable.

As the number of divisions of the spatial domain n increases, the size of M also increases. The range of eigenvalues increases correspondingly and the rank becomes difficult to compute numerically. More importantly, however, this also means that the system needs large values of the control input which may not be available in practice. The condition number of M, C_M, being the ratio of the largest to smallest singular values, is thus an indicator of the degree to which the system may be controlled with a bounded input. Of course, M is singular and not of full rank if C_M is infinite. Fig. 5 shows the effect of n on C_M. As n increases, it becomes increasingly difficult to take the system to a desired θ.

So far the air and water flow rates have been fixed. Fig. 6 shows the effect of varying air velocity on C_M, and it is not appreciable. On the other hand the water velocity is seen to have a stronger influence, as shown in Fig. 7. At the minimum C_M the system is the most controllable. Thus the condition number provides information on the best flow rates for control of the heat exchanger when the inlet water temperature is used as a manipulated variable.

4.3. Output controllability

There are many different possibilities of outputs that may be controlled. Some of the those that have practical use are the following.

(a) One example of an output of the system is the heat exchanger tube wall temperature distribution, for which

$$C = \mathrm{diag}[1, 0, 1, \ldots, 0] \in \mathbb{R}^{(2n-1)\times(2n-1)}.$$

The output controllability matrix N has the same size as C. However, N is not of full rank, so that this output is not controllable.

(b) Another is the outlet water temperature T_w^{out}, so that

$$C = [0, \ldots, 0, 1] \in \mathbb{R}^{1\times 2n-1},$$

for which

$$N = \left[0, \ldots, \frac{\dot{m}_w}{\rho_w \pi r_i^2 \Delta x}\left(\frac{\alpha_t}{\Delta x^2}\right)^{n-1} + \left(\frac{\dot{m}_w}{\rho_w \pi r_i^2 \Delta x}\right)^{n-1} \frac{4h_i}{\rho_w c_w D_i}, \ldots\right].$$

The output is controllable.

(c) A third example that is also of practical interest is the average outlet air temperature

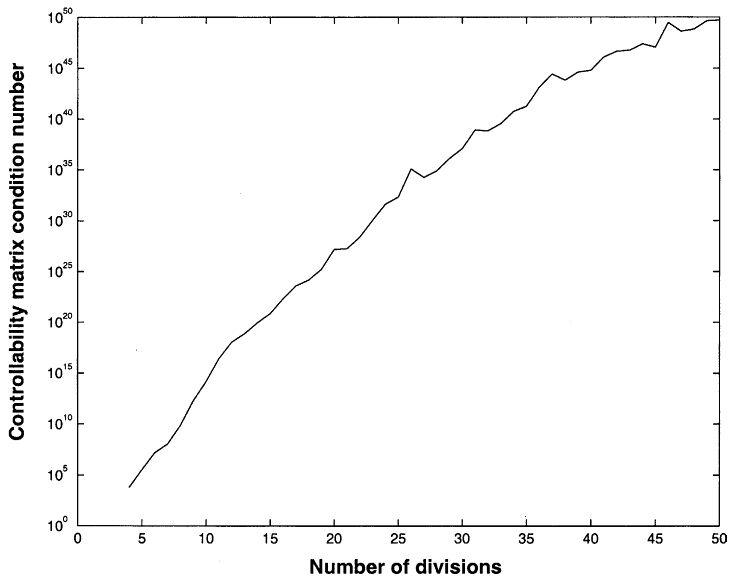

Fig. 5. The effect of the number of divisions on the condition number when water inlet temperature is the manipulated variable.

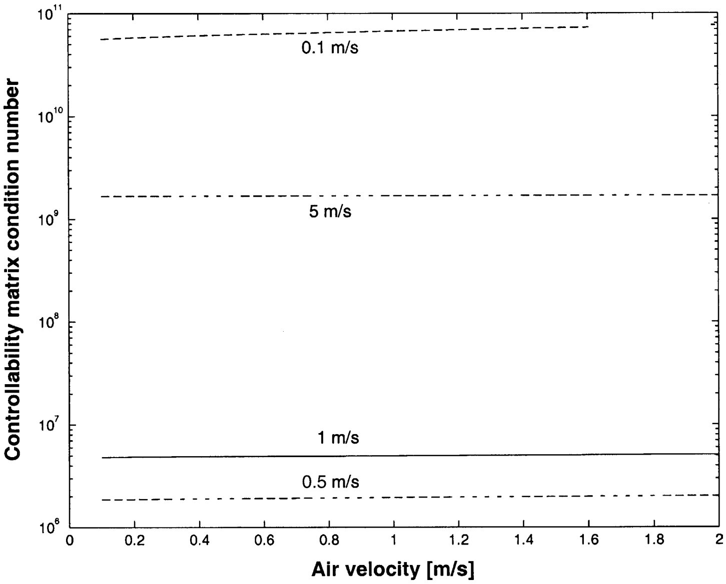

Fig. 6. The effect of air flow rate on the condition number when water inlet temperature is the manipulated variable; four different water velocities are indicated.

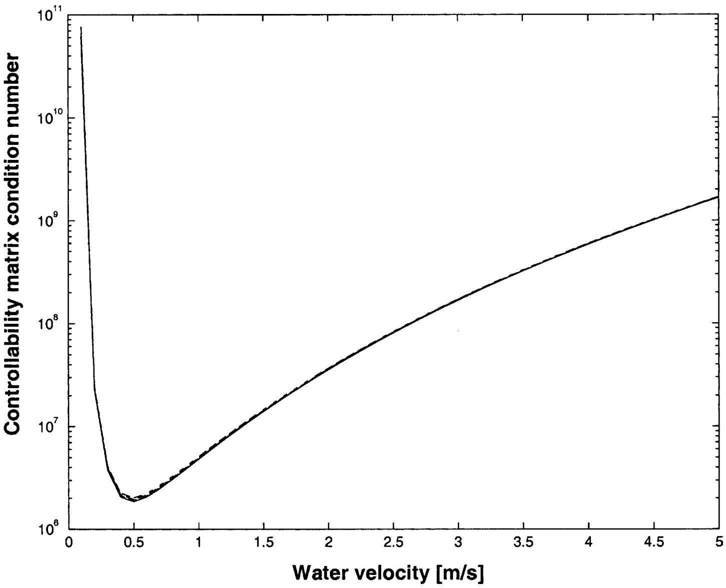

Fig. 7. The effect of water flow rate on the condition number when water inlet temperature is the manipulated variable; air flow is in the 0.1–2 m/s range.

$$\overline{T}_{\mathrm{a}}^{\mathrm{out}}(t) = \frac{1}{L}\int_0^L T_{\mathrm{a}}^{\mathrm{out}}(x,t)\,\mathrm{d}x, \tag{24}$$

where

$$T_{\mathrm{a}}^{\mathrm{out}}(x,t) = \frac{(1-a_5/2)T_{\mathrm{a}}^{\mathrm{in}} + a_5 T_{\mathrm{t}}}{1+a_5/2},$$

is used with the trapezoidal rule for integration. The matrix

$$C = \Delta x\left[\frac{1}{2},0,1,0,\ldots,1,0,\frac{1}{2}\right] \in \mathbb{R}^{1\times(2n-1)} \tag{25}$$

with $p = 1$. The output controllability matrix is

$$N = \Delta x\left[\frac{\alpha_t}{2\Delta x^2}, \ldots\right].$$

N has a rank equal to p, indicating that the system is output controllable.

5. Manipulated variable: air inlet temperature

5.1. State controllability

It is also common to have the inlet air temperature T_a^{in} as a manipulated variable. Matrix A is still as shown in Eq. (21) but

$$B = [a_3, 0, a_3, 0, \ldots, a_3]^T \in \mathbb{R}^{2n-1\times 1},$$

$$F = \left[\frac{\alpha_t}{\Delta x^2}T_w^{in}, \frac{\dot{m}_w}{\rho_w \pi r_i^2 \Delta x}T_w^{in}, 0, \ldots, 0\right]^T \in \mathbb{R}^{2n-1\times 1},$$

$$u = T_a^{in}.$$

With the transformation of Eq. (23), the governing equation can be reduced to the form of Eq. (5) and the controllability matrix defined by Eq. (8) can be computed. It is found that M has a full rank, indicating the system is controllable: by changing T_a^{in} any set of water and tube wall temperatures at a finite number of points can be reached in finite time. When different water and air velocity are used, a similar phenomenon occurs as in Section 4.2 when the water inlet temperature was the manipulated variable. The results of varying the air and water flow rates on C_M are shown in Figs. 8 and 9, respectively. Again the water flow rate is found to have a significant effect but not the air flow rate. There is an optimum water flow rate at which the system is the most controllable.

5.2. Output controllability

If the output is the average outlet air temperature defined by Eq. (24), it can be calculated with the C matrix in Eq. (25). The output controllability matrix is

$$N = \Delta x[(n-1)a_3, \ldots].$$

N has a rank equal to p, indicating that the system is output controllable.

6. Manipulated variable: water velocity

In this Section the objective is to control the outlet water temperature T_w^{out} by manipulating the water flow rate $\dot{m}_w$ while keeping constant the air flow rate $\dot{m}_a$, and the inlet air and water temperatures T_a^{in} and T_w^{in}, respectively. If the water velocity is used as a manipulated variable, the situation is entirely different from those treated before. The control problem is non-linear since the manipulated variable $\dot{m}_w$ appears as a product with the unknown temperature $T_w(x,t)$ in Eq. (14). The previously used linear controllability ideas cannot be directly applied in this situation. Furthermore, controllability of a linearized approximation does not imply controllability over the entire state space [13,14].

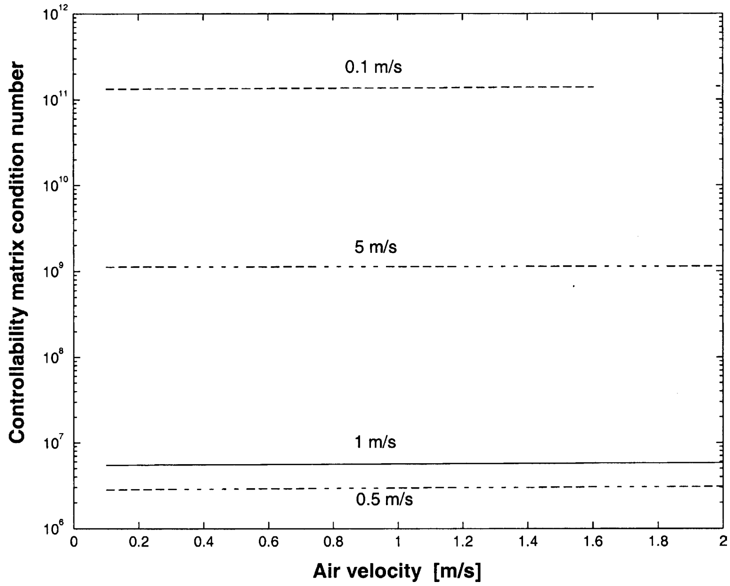

Fig. 8. The effect of air flow rate on the condition number when air inlet temperature is the manipulated variable; four different water velocities are indicated.

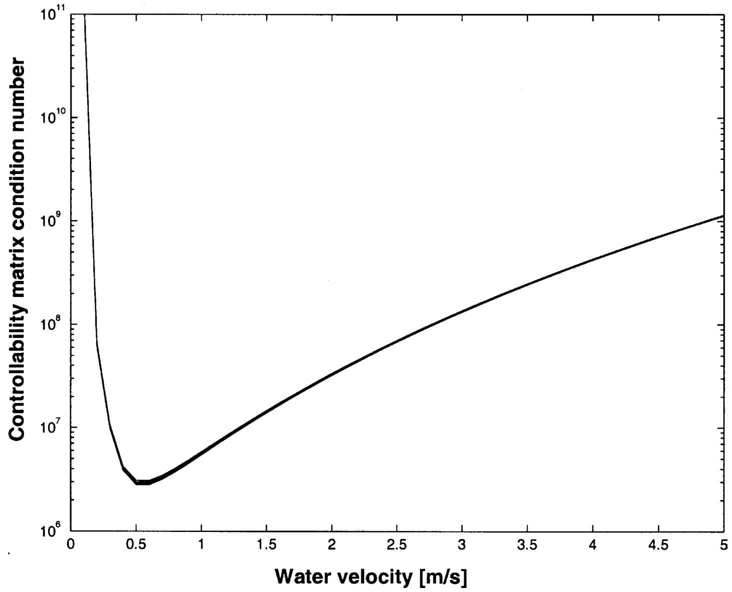

Fig. 9. The effect of water flow rate on the condition number when air inlet temperature is the manipulated variable; air flow is in the 0.1–2 m/s range.

Finding the range of T_w^{out} by solving Eqs. (12)–(15) turns out to be a difficult task. However, it is obvious that by manipulating the water velocity even over the entire range of positive real numbers one cannot reach all possible water outlet temperatures. Two steady-state extremes can be considered. When the water flow rate is small the advective term in the steady-state version of Eq. (14) is also small, so that $T_t = T_w$. Substituting this in steady Eq. (15) where the conduction along the tube wall is now negligible, we have $T_t = T_a$. Since this cannot satisfy the boundary conditions there is a thin boundary layer near the entrance $x = 0$. The water temperature at the outlet is $T_w^{out} = T_a^{in}$. Similarly at the other extreme, for large flow rates Eqs. (15) and (14) give $T_w^{out} = T_w^{in}$. Thus, in general, in the steady state T_w^{out} is between the two temperatures, T_a^{in} and T_w^{in}. Since this range of temperatures can be reached in the steady state, it follows that these states are controllable.

The arguments above are not valid for unsteady situations where the dynamics of the control system should be taken into account. However, one can invoke the laws of thermodynamics to assert that the local, instantaneous temperature at any point within the heat exchanger cannot be outside the (T_a^{in}, T_w^{in}) range. Thus T_w^{out} is controllable only within this range and is not globally controllable if $\dot{m}_w$ is the manipulated variable.

7. Conclusions

The controllability of cross-flow heat exchangers is investigated from a theoretical point of view. This property guarantees the ability of the heat exchanger to transfer a system from an initial to a final state. However, this does not imply that the system will stay there. State controllability is for the complete system while output controllability refers to the output only. A finite-difference approach has been introduced for the analysis of systems governed by a coupled set of PDEs. Two fundamentally different problems arise when the temperatures and the flow rates are used as a manipulated variables. The former is linear, and the latter non-linear.

Cases of controllability with respect to different manipulated variables were analyzed here. The heat exchanger was found to be less controllable for high air and water flow rates. There is also the issue of practical controllability that can be quantified on the basis of the condition number of the controllability matrix. It is found that there is an optimum water flow rate at which the heat exchanger is the most controllable.

In linear controllability it is assumed that the dependent variables are allowed to vary over a semi-infinite range. There are practical restrictions to this since the temperatures and flow rates can take on only positive values. Thus the constraints $T_w > 0$, $T_t > 0$, $\dot{m}_w > 0$ and $\dot{m}_a > 0$ impose restrictions on the range over which the heat exchanger is controllable. There is an even greater restriction on the manipulated variables; they cannot be varied in an infinite range even if positive. In this sense all the input variables are constrained: the inlet air and water temperatures are between certain bounds as are the air and water flow rates. This also reduces the controllability of the system. So, a system that is theoretically controllable may not practically be so because of these limitations on both the system and manipulated variable.

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